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# Some Representations of Solutions of Elliptic Equations

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The Dirichlet problem for the region of the plane inside closed smooth curve $C$ for second-order elliptic equations is considered. It is shown that under certain circumstances the solution $u$ can be written uniquely in the form $u(P) = \int_c F(P, Q)\, g(Q)\, ds_Q$, where $F(P, Q)$ is the fundamental solution of the elliptic equation, and $g \in L^2$ if the boundary value function $f$ is absolutely continuous with square integrable derivative ($f \in W$); and $u(P) = \rho(F(P, \cdot))$ where $\rho$ is a unique bounded linear functional on $W$ if $f \in L^2$. These representations are valid in the exterior of $C$ also. As special cases with slight modifications, the exterior Dirichlet problems for the Helmholtz and Laplace equations are mentioned.

It is shown also that if kernel $F(P', Q)$, with $P'$ and $Q$ on $C$, has a complete set of eigenfunctions $\{\psi_k(P')\}$ then $u(P)$ can be expanded in a series of their extensions $\{\psi_k(P)\}$, where $\psi_k(P) = \lambda_k \int_c F(P, Q)\, \psi_k(Q)\, ds_Q$.

This paper concerns the representation of solutions of Dirichlet's problem for linear homogeneous elliptic second-order partial differential equations. These representations are of two types: (1) the solution in terms of a single-layer distribution, and (2) the solution as a series of eigenfunctions.

The ideas are carried out in detail for Laplace's equation. With the fundamental solution $G(P, Q) = \log |P - Q|$, the solution $u(P)$ of the interior problem can be written (under certain circumstances) as

$$u(P) = -\int_0^{2\pi} G(P, Q(\tau))\, g(\tau)\, d\tau,$$

with $g \in \mathscr{L}^2$ if the boundary values are continuously differentiable; and as

$$u(P) = -\lim_{n\to\infty} \int_0^{2\pi} G(P, Q(\tau))\, g_n(\tau)\, d\tau,$$

with $g_n \in \mathscr{L}^2$, if the boundary values are in $\mathscr{L}^2$. The "eigenfunction" expansions follow from these representations. (As Jaswon [6] indicates, the formulation of the exterior problem may require some modification. This is discussed briefly in Section 7.)

Similar representations are possible for other elliptic equations using the fundamental solution. Wolfe [1] has done some of this for the reduced wave equation for the exterior of a slit in the plane; much of Section 2 and 3 are based on his work. Karp and Shamma [2] have considered "eigen-function" expansions as a starting point for their investigation of the asymptotic properties of the eigenfunctions associated with various problems.

1. $C$ is a closed curve without double-points in the $xy$-plane, given parametrically by $x = x(t)$, $y = y(t)$, with both functions periodic with period $2\pi$, twice continuously differentiable, and $\dot{x}^2 + \dot{y}^2 \neq 0$. ($\dot{g} = dg/dt$.) The problem is to find a harmonic function $u(x, y)$ inside $C$ (or outside $C$) taking given values $f(t)$ on $C$ and representable in the form

$$u(P) = \int_0^{2\pi} -g(\tau) \log | P - Q(\tau)| \, d\tau, \tag{1}$$

where $P$ is the point $(x, y)$, $Q(\tau)$ is $(x(\tau), y(\tau))$, $| P - Q |$ is the distance between $P$ and $Q$. When $P$ is on $C$, $u(P(t)) = f(t)$, and (1) becomes

$$f(t) = - \int_0^{2\pi} g(\tau) \log | P(t) - Q(\tau)| \, d\tau. \tag{2}$$

The problem then becomes one of finding for $f$ in a certain class of functions a solution $g$ of (2) in another class such that $u(P)$ as defined by (1) tends to $f(t)$ as $P$ tends to $P(t)$ on $C$.

As preliminaries for this investigation we note that with $0 \leqslant \rho < 1$, $\Phi_k(\tau) = e^{ik\tau}/(2\pi)^{1/2}$,

$$\log | \rho e^{it} - e^{i\tau} | = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} (2\pi\rho^{|k|}/| k |) \, \Phi_k(t) \, \overline{\Phi_k(\tau)} \tag{3}$$

and

$$\log | \sin((t - \tau)/2)| = -\log 2 + \sum_{k=-\infty}^{\infty} (2\pi/| k |) \, \Phi_k(t) \, \overline{\Phi_k(\tau)}. \tag{4}$$

Thus, $\{\Phi_k\}$, $k = 0, \pm 1, \pm 2, \ldots$ form a complete orthonormal system of eigenfunctions of (4) in $\mathscr{L}^2(0, 2\pi)$ with corresponding eigenvalues $| k |/2\pi$, $k \neq 0$, and $-1/2\pi \log 2$, if $k = 0$.

The class $W$ of functions consists of those absolutely continuous functions $f$ on $[0, 2\pi]$ with derivative $f'$ in $\mathscr{L}^2(0, 2\pi)$ and with period $2\pi$. This class is a Hilbert space with norm defined by

$$(\| f \|_W)^2 = \int_0^{2\pi} (| f |^2 + | f' |^2) \, d\tau.$$

If $f(\tau) = \sum_{k=-\infty}^{\infty} c_k \Phi_k(\tau)$, then

$$\|f\|_W^2 = \sum_{k=-\infty}^{\infty} |c_k|^2(1 + k^2).$$

**2.** In this section we show that if $f \in W$, then there is a unique $g \in \mathscr{L}^2(0, 2\pi)$ satisfying (2) under certain conditions.

LEMMA 1. *If* $\sum_{-\infty}^{\infty} k^2 |c_k|^2 < \infty$ *then* $f(\tau) = \sum_{-\infty}^{\infty} c_k \Phi_k(\tau)$ *is in* $W$, *and conversely.*

*Proof.* The function $q(\tau) = \sum_{-\infty}^{\infty} ikc_k \Phi_k(\tau)$ is in $\mathscr{L}^2(0, 2\pi)$. By the general theory of the Fourier series, this last series may be integrated term by term from 0 to $x$. $(\sum |c_k|)^2 \leqslant \sum k^2 |c_k|^2 \sum 1/k^2$ by the Schwarz inequality so $\sum c_k$ is convergent. Thus

$$\int_0^x q(\tau)\, d\tau + \sum_{-\infty}^{\infty} c_k/(2\pi)^{1/2} = \sum_{-\infty}^{\infty} c_k \Phi_k(x).$$

Thus, $f \in AC$ on $[0, 2\pi]$ and $f' = q$ a.e.

For future ease of reference, the following notation of kernels and operators will be used.

$$\begin{aligned} G(P, Q) &= \log |P - Q|, \\ K(t, \tau) &= \log |\sin((t - \tau)/2)|, \\ R(P, Q) &= \log\{|P - Q|/|\sin((t - \tau)/2)|\} \end{aligned} \tag{5}$$

$$\mathbf{G}g = \int_0^{2\pi} G(P(t), Q(\tau))\, g(\tau)\, d\tau,$$

$$\mathbf{G}(P)\, g = \int_0^{2\pi} G(P, Q(\tau))\, g(\tau)\, d\tau,$$

$$\mathbf{K}g = \int_0^{2\pi} K(t, \tau)\, g(\tau)\, d\tau,$$

$$\mathbf{R}g = \int_0^{2\pi} R(P(t), Q(\tau))\, g(\tau)\, d\tau,$$

$$\mathbf{G} = \mathbf{K} + \mathbf{R}.$$

Under the hypothesis on curve $C$, $R(P(t), Q(\tau))$ is a continuously differentiable function of $t$ and $\tau$. Equation (2) can be written in the operator form

$$-f = \mathbf{G}g = \mathbf{K}g + \mathbf{R}g \tag{6}$$

LEMMA 2. **K** *is a bounded linear transformation mapping* $\mathscr{L}^2$ *one-to-one onto* $W$. $\mathbf{K}^{-1}$ *is bounded on* $W$.

*Proof.* Replacing $\log|\sin(t-\tau)/2|$ by the series in (4), and using $g = \sum c_k\Phi_k$, $f = \sum b_k\Phi_k$ in $f = Kg$, yields from (5)

$$\sum b_k\Phi_k = -(c_0 2\pi \log 2)\,\Phi_0 + \sum_{k\neq 0} (2\pi/|k|)\, c_k\Phi_k\,,$$

from which

$$b_0 = -2\pi c_0 \log 2, \qquad b_k = 2\pi c_k/|k|. \tag{7}$$

Thus $\sum |c_k|^2 < \infty$ implies $\sum k^2 |b_k|^2 < \infty$ and conversely, establishing the one-to-one onto. $\|\mathbf{K}g\| = \|f\|_W \leqslant 4\pi \|g\|_{\mathscr{L}^2}$ from (7). $\mathbf{K}^{-1}$ is bounded on $W$ by Banach's inverse theorem (or by direct computation).

LEMMA 3. **R** *is a compact operator from* $\mathscr{L}^2$ *into* $W$.

*Proof.* Let $\Omega$ be a bounded set (in $\mathscr{L}^2$ norm) in $\mathscr{L}^2$. $\{\mathbf{R}g : g \in \Omega\}$ satisfy a common Lipschitz condition. Indeed, this family all $\in C'$ with uniformly bounded derivative. Moreover, the derivatives form, an equicontinuous family, the modulus of continuity depending on that of $(\partial/\partial s)\, R(s, t)$ and the bound on $\Omega$. Hence $\mathbf{R}\Omega$ contains a subsequence such that the functions and their derivatives are uniformly convergent. Thus the limit is in $C'$ and the subsequence converges in the norm of $W$. Thus **R** maps bounded sets in $\mathscr{L}^2$ into relatively sequentially compact sets in $W$.

THEOREM 1. *If* $\mathbf{G}v = 0$ *has only the trivial solution in* $\mathscr{L}^2(0, 2\pi)$ *(e.g., if* **G** *has a complete set of eigenfunctions in* $\mathscr{L}^2$*), then* (6) *has a unique solution* $g$ *in* $\mathscr{L}^2$ *for any* $f$ *in* $W$. $\mathbf{G}^{-1}$ *is bounded on* $W$. $u(P)$ *defined by* (1) *is continuous throughout the plane and satisfies* $\Delta u = 0$ *for* $P$ *off* $C$.

*Proof.* Equation (6) has a solution in $\mathscr{L}^2$ if and only if

$$-\mathbf{K}^{-1}f = g + \mathbf{K}^{-1}\mathbf{R}g \tag{8}$$

does. $\mathbf{K}^{-1}\mathbf{R}$ is a compact operator on $\mathscr{L}^2$ into $\mathscr{L}^2$ since $\mathbf{K}^{-1}$ is bounded (by Lemma 2) and **R** is compact. By the Riesz theory of compact operators, (8) is solvable, and uniquely so, if $g = -\mathbf{K}^{-1}\mathbf{R}g$ has only the trivial solution in $\mathscr{L}^2$. But this equation implies $\mathbf{K}g + \mathbf{R}g = 0$. By hypothesis, this last equation has only the trivial solution. Therefore, (8) has a unique solution in $\mathscr{L}^2$ for each $f$ in $W$. $\mathbf{G}^{-1}$ is bounded by Banach's inverse theorem.

That $u(P)$ is harmonic for $P$ off $C$ is verified by differentiation under the integral sign. The boundary behavior will be discussed in Section 4.

*Remark.* The hypothesis that $\mathbf{G}v = 0$ has only the trivial solution is essential. For $C$, the unit circle, this equation has the solution $v = 1$ and, consequently, (6) is not solvable for $f$ equal to a nonzero constant or, indeed, for any $f$ for which

$$\int_0^{2\pi} f(t)\, dt \neq 0.$$

Jaswon [6] shows that for any family of similarity curves, $Gv = 0$ has a nontrivial solution for just one of them.

3. In this section Theorem 1 is extended to include a wider class of $f$ and, correspondingly, a wider class of $g$.

DEFINITION. $W^*$ denotes the conjugate space of $W$, i.e., $W^*$ is the space of all bounded linear functionals on $W$. $M$ denotes a linear manifold in $W^*$. An element $T$ belongs to $M$ if and only if for some $n$

$$T(f) = \sum_{k=-n}^{n} c_{-k} b_k ,$$

where

$$f(\tau) = \sum_{-\infty}^{\infty} b_k \Phi_k(\tau)$$

is an arbitrary element in $W$. $M^*$ denotes the closure of $M$ in $W^*$. Alternatively, $T$ belongs to $M$ if and only if for some $n$,

$$T(f) = \int_0^{2\pi} g_n(\tau) f(\tau)\, d\tau,$$

where

$$g_n(\tau) = \sum_{k=-n}^{n} c_k \Phi_k(\tau).$$

LEMMA 4. *An element $T$ of $W^*$ belongs to $M^*$ if and only if $T(f) = \sum_{-\infty}^{\infty} c_{-k} b_k$, where $f = \sum b_k \Phi_k$ and $\sum | c_k/k |^2 < \infty$. If $r_n^2 = | c_0 |^2 + \sum_{-n}^{n} | c_k/k |^2$ and $r_n \to r$ as $n \to \infty$, then $r/2 \leqslant \| T \|_{W^*} \leqslant r$.*

*Proof.* Let

$$T_n(f) = \sum_{k=-n}^{n} c_{-k} b_k .$$

Then

$$|T_n(f) - T_m(f)|^2 = \left| \sum_{|k|=n}^{m} (c_{-k}/k)(kb_k) \right|^2$$

$$\leqslant \sum_{|k|=n}^{m} | c_k/k |^2 \sum_{|k|=n}^{m} | kb_k |^2$$

$$\leqslant \sum_{|k|=n}^{m} | c_k/k |^2 \| f \|_W^2 . \tag{9}$$

If $\sum | c_k/k |^2 < \infty$, then $T_n(f)$ converges by the Cauchy criterion. By a similar computation, $| T_n(f)|^2 \leqslant r_n^2 \| f \|_W^2$. Also, $(| T_n(f_n)|/\| f_n \|)^2 \geqslant r_n^2/2$ if $f_n = \bar{c}_0 + \sum_{-n}^{n} (\bar{c}_{-k}/k^2) \Phi_k'$. Therefore, for fixed $n$, $\| T_n \|_{W^*} \geqslant r_n/2$. The bound on norm $T$ follows on letting $n$ tend to infinity. From (9) it is clear that

$$\sup_{\|f\|_W=1} | T_n(f) - T(f)| \to 0 \text{ as } n \to \infty,$$

i.e., $\| T_n - T \|_{W^*} \to 0$. For the "only-if" part, suppose $\| S_n - T_n \|_{W^*} \to 0$, $S_n \in M$. Then $S_n(f)$ converges for every $f$ in $W$. In particular

$$\lim_{n\to\infty} S_n(f) = \lim \sum_{-n}^{n} c_{-k}^{(n)} b_k = \sum_{-m}^{m} c_{-k} b_k = T(f)$$

if

$$f(\tau) = \sum_{-m}^{m} b_k \Phi_k(\tau).$$

Therefore,

$$\sum c_{-k} b_k = \sum (c_{-k}/k)(kb_k)$$

is convergent for every sequence of $\{kb_k\}$ which is absolutely square convergent. From Banach's principle of uniform boundedness it follows that $\{c_k/k\}$ is absolutely square convergent.

Hence any element of $M^*$ is the pointwise limit of

$$\rho_n(f) = \int_0^{2\pi} g_n(\tau) f(\tau)\, d\tau = \sum_{-n}^{n} c_{-k} b_k , \tag{10a}$$

where

$$g_n(\tau) = \sum_{-n}^{n} c_k \Phi_k(\tau), \qquad \sum_{-\infty}^{\infty} | c_k/k |^2 < \infty. \tag{10b}$$

Equation $f_n = Kg$ can be solved for $f_n = \sum_{-n}^{n} b_k \Phi_k$ using formulas (7), yielding a solution of form (10b). The operation $\mathbf{K}g_n$ can be extended to $M^*$ by limits, since

$$\begin{aligned}\| \mathbf{K}g_n \|_{\mathscr{L}^2}^2 &= 4\pi^2 \left\| -c_0\Phi_0 \log 2 + \sum_{k=-n}^{n} c_k \Phi_k / | k | \right\|_{\mathscr{L}^2}^2 \\ &\leqslant 8\pi^2 \left( | c_0 |^2 + \sum_{-n}^{n} | c_k/_k |^2 \right) \\ &\leqslant 16\pi^2 \| \rho_n \|_{M^*}^2\end{aligned}$$

with $\rho_n$ and $g_n$ as in (10a) and (10b).

LEMMA 5. ***K** is a bounded linear transformation on $M^*$ and is one-to-one onto $\mathscr{L}^2$. $\mathbf{K}^{-1}$ is bounded on $\mathscr{L}^2$.*

*Proof.* Proof that **K** is bounded comes from the above discussion. If $f \in \mathscr{L}^2$, then $\sum | b_k |^2 < \infty$. Formulas (7) show that consequently $\sum | c_k/k |^2 < \infty$, and so the solutions $g_n$ of $f_n = \mathbf{K}g$ converge in $M^*$ to some $\rho$. $\mathbf{K}g_n$ converges in the mean of order two to $f$. The boundedness of $\mathbf{K}^{-1}$ can be obtained by direct calculation or by invoking Banach's inverse mapping theorem.

LEMMA 6. *If $H(s, t) \in C'$, then the operator* **H** *defined by*

$$h(s) = \mathbf{H}\rho = \lim_{n\to\infty} \int_0^{2\pi} H(s, t)\, g_n(t)\, dt,$$

*and $g_n$ given by* (10b), *is a compact operator on $M^*$ into $\mathscr{L}^2$.*

*Proof.* Since $H(s, t) \in C'$, $\mathbf{H}\rho$ is interpreted as the bounded linear functional $\rho$ evaluated at $H$ regarded as a function of $t$, viz., $\rho(H) = \mathbf{H}\rho$,

$$| \mathbf{H}\rho | = | \rho(H(s, \cdot))| \leqslant \| \rho \|_{M^*} \| H(s, \cdot)\|_W ,$$

$$\| H(s, \cdot)\|_W^2 = \int_0^{2\pi} (| H(s, t)|^2 + |(\partial/\partial t)\, H(s, t)|^2)\, dt.$$

If $\Omega$ is a bounded set in $M^*$, say $\sup_{\rho\in\Omega} \| \rho \| = \alpha$, then

$$\begin{aligned}| h(s)| &= | \rho(H(s, \cdot))| \leqslant \alpha \| H(s, \cdot)\|_W , \\ | h(s_1) - h(s_2)| &= | \rho[H(s_1 , \cdot) - H(s_2 , \cdot)]| \\ &\leqslant \alpha \| H(s_1 , \cdot) - H(s_2 , \cdot)\|_W .\end{aligned}$$

Thus $\{h(s): h = \mathbf{H}\rho, \rho \in \Omega\}$ form an equicontinuous, uniformly bounded family of functions. By Arzela's theorem, this family is relatively sequentially compact in $\mathscr{L}^2$.

THEOREM 2. *If* $\mathbf{G}\sigma = 0$ *has only the trivial solution* $\rho$ *in* $M^*$, (*e.g., if* $\mathbf{G}$ *has a complete set of eigenfunctions in* $\mathscr{L}^2(0, 2\pi)$), *then* $-f = \mathbf{G}\rho$ *has a unique solution in* $M^*$ *for each* $f$ *in* $\mathscr{L}^2(0, 2\pi)$. $\mathbf{G}^{-1}$ *is bounded on* $\mathscr{L}^2$; $u(P)$ *defined by* $u(P) = -\mathbf{G}(P)\rho$ *satisfies* $\Delta u = 0$ *for* $P$ *off* $C$; $u(P)$ *is continuous throughout the plane if* $f$ *is continuous, and approaches* $f(t)$ *as* $P$ *tends to* $P(t)$ *along the normal to the curve at* $P(t)$ *for almost all* $t$ *if* $f \in \mathscr{L}^2$.

*Proof.* The proof of existence follows as in the proof of Theorem 1. If $\rho$ is the solution, there is a sequence $\rho_n$, $g_n$ as in (10a) and (10b), with $\rho_n$ converging in $M^*$ to $\rho$.

$$u_n(P) = -\mathbf{G}(P)\,\rho_n = -\int_0^{2\pi} g_n(\tau) \log | P - Q(\tau)| \, d\tau \tag{11}$$

is harmonic off $C$. If $P$ is off $C$,

$$| u_n(P) - u(P)| \leqslant \| \rho_n - \rho \|_{M^*} \| \log | P - Q | \|_W .$$

Thus $u(P)$ is the uniform limit of $u_n(P)$, and so is harmonic. The boundary behavior is left to Section 4.

It is not trivial that $\mathbf{G}\sigma = 0$ has only the trivial solution in $M^*$ if $\mathbf{G}$ has a complete set of eigenfunctions in $\mathscr{L}^2$. The proof results from the following lemma.

LEMMA 7. *If* $\{\psi_k\}$ *is a complete system of eigenfunctions of* $\mathbf{G}$ *in* $\mathscr{L}^2$, *then it is a complete system in* $W$.

*Proof.* $\mathbf{G}\psi_k$ is in $W$ by Section 2, and so $\{\psi_k\}$ is in $W$ since $\psi_k = \lambda_k \mathbf{G}\psi_k$. It is sufficient to prove that $\{\psi_k\}$ has the closure property in $W$. Let $f$ be an arbitrary fixed element in $W$. By Section 2 there is a $g$ in $\mathscr{L}^2$ such that $f = \mathbf{G}g$. Since $\{\psi_k\}$ are complete in $\mathscr{L}^2$ they have the closure property in $\mathscr{L}^2$ and so there exists a sequence $h_n$ in $\mathscr{L}^2$,

$$h_n = \sum_{k=1}^{n} a_k(n)\, \psi_k ,$$

with $\| h_n - g \|_{\mathscr{L}^2} \to 0$. Since $\mathbf{G}$ is a bounded operator on $\mathscr{L}^2$,

$$\| f_n - f \|_W \leqslant \| G \| \, \| h_n - g \|_{\mathscr{L}^2} ,$$

where

$$f_n = \sum_{k=1}^{n} \lambda_k a_k(n)\, \psi_k .$$

Thus $\{\psi_k\}$ has the closure property in $W$.

LEMMA 8. *If* $\mathbf{G}\rho = 0$, $\rho$ *in* $M^*$, *then* $\rho = 0$ *if* $\mathbf{G}$ *has a complete set of eigenfunctions in* $\mathscr{L}^2$.

*Proof.* With notation as in Lemma 7, $\psi_k$ is in the domain of $\rho$. Thus, with $g_n$, $\rho_n$ as in (10a) and (10b), $\rho_n \to \rho$ in $M^*$,

$$\begin{aligned}\rho(\psi_k) &= \lambda_k \rho(\mathbf{G}\psi_k)\\ &= \lambda_k \lim_{n\to\infty} \int_0^{2\pi} \left[\int_0^{2\pi} \log|P(t) - Q(\tau)|\, \psi_k\, d\tau\right] g_n(t)\, dt\\ &= \lambda_k \lim_{n\to\infty} \int_0^{2\pi} \left[\int_0^{2\pi} \log|P(t) - Q(\tau)|\, g_n(t)\, dt\right] \psi_k\, d\tau\\ &= \lambda_k \int_0^{2\pi} (\mathbf{G}\rho)\, \psi_k\, d\tau = 0.\end{aligned}$$

The limit is placed under the integral sign since $\mathbf{G}g_n \to \mathbf{G}\rho$ in $\mathscr{L}^2$, since $g_n \to \rho$ in $M^*$. Thus $\rho(\psi_k) = 0$ for every member of the complete set $\{\psi_k\}$ in $W$. Therefore $\rho = 0$.

**4.** To examine boundary behavior it is convenient to introduce at each point on the boundary a local coordinate system.

LEMMA 8. *If* $x = \Phi(s, t)$, $y = \psi(s, t)$ *is a class* $C'$ *mapping in a neighborhood containing* $(s_1, t_1)$, *then*

$$\left|\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}\right|^2 = \left|J\begin{bmatrix}\Delta s\\ \Delta t\end{bmatrix}\right|^2 + \epsilon \left|\begin{bmatrix}\Delta s\\ \Delta t\end{bmatrix}\right|^2,$$

*where* $\Delta x = x - x_1 = \Phi(s, t) - \Phi(s_1, t_1)$, $\Delta y = y - y_1 = \psi(s, t) - \psi(s_1, t_1)$, *etc., and* $\epsilon \to 0$ *as* $(\Delta s)^2 + (\Delta t)^2 \to 0$, *and*

$$J = \begin{bmatrix}\Phi_s & \Phi_t\\ \psi_s & \psi_t\end{bmatrix}$$

*evaluated at* $(s_1, t_1)$.

The proof is obvious and will be omitted.

If curve $C$ is given by $\vec{r}(t)$, then a normal to $C$ at $t$ is $\vec{n}(t) = \dot{y}(t)\,\hat{\imath} - \dot{x}(t)\,\hat{\jmath}$, where $\cdot$ denotes $d/dt$. The transformation

$$\begin{aligned}x\hat{\imath} + y\hat{\jmath} &= \Phi(s, t)\,\hat{\imath} + \psi(s, t)\,\hat{\jmath}\\ &= \vec{r}(t) + s\vec{n}(t)\\ &= [x(t) + s\dot{y}(t)]\,\hat{\imath} + [y(t) - s\dot{x}(t)]\,\hat{\jmath} \qquad (12)\end{aligned}$$

induces a local coordinate system at a point $P(t_0)$ on the curve $C$, the grid corresponding to curves on which $s$ is constant and those on which $t$ is constant. The Jacobian at $s = 0$ (i.e., on the curve $C$) is $\dot{x}^2 + \dot{y}^2 > 0$. With $P = (x, y)$ given by (12),

$$| P - Q(\tau)|^2 = \left|\begin{bmatrix} \dot{y}(t) & \dot{x} \\ -\dot{x} & \dot{y} \end{bmatrix}\begin{bmatrix} s \\ t - \tau \end{bmatrix}\right|^2 + \epsilon^* \left|\begin{bmatrix} s \\ t - \tau \end{bmatrix}\right|^2$$

where $\epsilon^* \to 0$ as $\omega^2 = (t - \tau)^2 + s^2 \to 0$. Thus if $\delta^* > 0$ is sufficiently small there are positive constants $k_1$ and $k_2$ such that

$$k_1\omega^2 \leqslant | P - Q(\tau)|^2 \leqslant k_2\omega^2 \tag{13}$$

if $\omega < \delta^*$.

THEOREM 3. *If $g \in \mathscr{L}^2$, $u$ defined by* (1) *is continuous throughout space.*

*Proof.* Given $\epsilon > 0$, there is a $\delta > 0$ such that

$$\int_{t-\delta}^{t+\delta} \log^2 | t - \tau | \, d\tau < (\epsilon^2/4 \| g \|^2).$$

Thus

$$\int_{t-\delta}^{t+\delta} | g(\tau) \log(\omega^2 + h^2)| \, d\tau < \epsilon/2$$

for all $h$ and $s$ sufficiently small. Thus

$$\int_{t-\delta}^{t+\delta} | g(\tau) \log(| P - Q(\tau)|^2 + h^2)| \, d\tau < (\epsilon/2)\, k_3 \tag{14}$$

for all $h$ sufficiently small, by use of (13) with $k_3$ depending on $k_1$ and $k_2$.

$$\begin{aligned} u_h(P) &= - \int_0^{2\pi} \tfrac{1}{2}g(\tau) \log(| P - Q(\tau)|^2 + h^2)| \, d\tau \\ &= - \int_0^{t-\delta} - \int_{t-\delta}^{t+\delta} - \int_{t+\delta}^{2\pi} . \end{aligned} \tag{15}$$

$u_h(P)$ is clearly a continuous function of $x$ and $y$. That $u_h$ converges uniformly to $u$ as $h \to 0$ follows from (14) and (15). Therefore, $u$ is continuous near and on $C$.

LEMMA 9. *If $q(\alpha, \tau)$ is periodic with period $2\pi$ in $\tau$ ($\alpha$ in a neighborhood of 1), is in class $C^2$ in $\tau$, vanishes only when $\tau = 0$ and with precisely the order $\tau^2$, and*

$$\lim_{\alpha\to 1} q^{(i)}(\alpha, \tau) = q^{(i)}(1, \tau), \qquad i = 0, 1, 2 \qquad (q^{(i)} = d^i q/d\tau^i),$$

*with the convergence bounded, then*

$$h(\alpha, \tau) = ((\alpha - 1)^2 + q(\alpha, \tau))/((\alpha - 1)^2 + 4\alpha \sin^2 \tau)$$

*has* $\lim_{\alpha\to 1} h^{(i)}(\alpha, \tau) = h^{(i)}(1, \tau)$, $i = 0, 1$; $\tau \neq 0$, *and the convergence is bounded.*

*Proof.* $\tau$ near zero poses the only difficulty and the necessary estimates on $q(\alpha, \tau)$, etc., are obtained from

$$q(\alpha, \tau) = \int_0^\tau (\tau - t)\, q''(\alpha, t)\, dt.$$

LEMMA 10. *If* $\rho \in M^*$ *and* $\rho_n$ *as in* (10a) *and* (10b) *converges to* $\rho$ *in* $M^*$, *then*

$$\lim_{\alpha\to 1^-} \lim_{n\to\infty} \int_0^{2\pi} g_n(\tau) \log | \alpha e^{it} - e^{i\tau} |\, d\tau = \underset{n\to\infty}{\text{l.i.m.}} \int_0^{2\pi} g_n(\tau) \log | e^{it} - e^{i\tau} |\, d\tau$$

*for almost all* $t$.

*Proof.* According to (3), the right side is a square integrable function $f(t)$ and the left side is the Abel sum of its Fourier series. These are equal at almost all $t$, in particular at those $t$ which are points of the Lebesgue set of $f$. If $f$ is continuous, the left side converges uniformly to $f$.

THEOREM 4. *If* $\rho \in M^*$ *is the solution of* (6) *for* $f \in L^2$, *and* $\rho_n$ *as in* (10a) *and* (10b) *converges to* $\rho$ *in* $M^*$ *then* $u(P) = \lim_{n\to\infty} u_n(P)$ *as defined by* (11) *converges to* $f(t)$ *for almost all* $t$ *if* $P$ *tends to* $P(t)$ *along the normal to* $C$ *at* $P(t)$. *If* $f$ *is continuous the convergence is uniform, and consequently* $u(P)$ *is continuous in the whole plane.*

*Proof.* With Eq. (12), and $P(s, t) = (x, y)$,

$$\begin{aligned} | P(s, t) - Q(\tau)|^2 = {} & | P(t) - Q(\tau)|^2 \\ & + 2s[\dot{y}(x(t) - x(\tau)) - \dot{x}(y(t) - y(\tau))] \\ & + s^2(\dot{x}^2 + \dot{y}^2). \end{aligned}$$

With $\alpha - 1 = s(\dot{x}^2 + \dot{y}^2)^{1/2}$, the first two terms can be grouped as $q(\alpha, \tau^*)$ satisfying the hypotheses of Lemma 9, with $\tau = t + \tau^*$. Thus

$$N(s, \tau) = | P(s, t) - Q(\tau)|/| \alpha e^{it} - e^{i\tau} |$$

is always positive and has the properties of $h$ in Lemma 9 for $s \to 0$. $\log N(s, \tau) \in C'$ in $\tau$ for each $s$ and is bounded along with the derivative for all $s$ near zero. Also lim as $s \to 0$ of $\log N(s, \tau)$ and its derivative converge pointwise to $\log N(0, \tau)$ and its derivative, respectively, for $\tau \neq 0$. Consequently,

$$\lim_{n\to\infty} \int_0^{2\pi} g_n(\tau) \log N(s, \tau)\, d\tau = N^*(s)$$

is continuous at $s = 0$. For

$$\begin{aligned} | N^*(s) - N^*(0) | &= | \rho(\log N(s, \tau) - \log N(0, \tau)) | \\ &\leqslant \| \rho \|_{M^*} \| \log N(s, \tau) - \log N(0, \tau) \|_{W}, \end{aligned}$$

and the last $W$-norm tends to zero as $s \to 0$, as can be seen by writing the integrals and using the bounded convergence to put the limit under the integral sign. Thus

$$u(P) = -\lim_{n\to\infty} \int_0^{2\pi} g_n(\tau) \log | \alpha e^{it} - e^{i\tau} |\, d\tau - \lim_{n\to\infty} \int_0^{2\pi} g_n(\tau) \log N(s, \tau)\, d\tau.$$

Taking lim as $s \to 0^-$ $(\alpha \to 1^-)$,

$$\lim_{P\to P_0} u(P) = -\lim_{n\to\infty} \int_0^{2\pi} g_n(\tau) \log | e^{it} - e^{i\tau} |\, d\tau - N^*(0)$$

for almost all $t$ using Lemma 10. But the right side is just $-\mathbf{G}\rho = f$. If $f$ is continuous the convergence is uniform by the observation in Lemma 10, since the results about $N^*$ are uniform over $t$. The continuity of $u(P)$ at points $P$ on the curve comes from the uniform approach and the usual triangle argument.

5. In this section certain eigenfunction expansions for the solution $u(P)$ of the Dirichlet problem are given.

If $\psi_k$ is an eigenfunction in $\mathscr{L}^2$ of $-G$, i.e., $\psi_k = -\lambda_k \mathbf{G}\psi_k$, then $\psi_k(P)$ is defined by

$$\psi_k(P) = \lambda_k \int_0^{2\pi} - \log | P - Q(\tau) |\, \psi_k(\tau)\, d\tau. \tag{16}$$

By the results of Sections 2 and 4, $\psi_k(P)$ is continuous throughout space and harmonic off $C$.

THEOREM 5. *Under the hypotheses of Theorem* 1, *if* $\{\psi_k\}$, $k = 1, 2,\ldots$, *is a complete orthonormal set of eigenfunctions of* $-\mathbf{G}$ *in* $\mathscr{L}^2$, *then*

$$u(P) = \sum_{k=1}^{\infty} a_k \psi_k(P), \tag{17}$$

*where*

$$a_k = \int_0^{2\pi} f(t)\, \psi_k(t)\, dt. \tag{18}$$

*The series is uniformly convergent in every compact region of the plane. Repeated termwise differentiation of the series is valid off C with the resulting series uniformly convergent in any compact region not intersecting C.*

*Proof.* Let $g$ be the solution of $f = -\mathbf{G}g$. Since $g \in \mathscr{L}^2$,

$$g(\tau) = \text{l.i.m.}\, g_n\,, \qquad g_n = \sum_{k=1}^{n} c_k\psi_k(\tau),$$

$$\begin{aligned} | u(P) - u_n{}^*(P)|^2 &= \left| \int_0^{2\pi} G(P, \tau)[g(\tau) - g_n(\tau)]\, d\tau \right|^2 \\ &\leqslant \int_0^{2\pi} G^2(P, \tau)\, d\tau \,\| g - g_n \|^2. \end{aligned} \tag{19}$$

That $u_n{}^*(P)$ converges uniformly to $u$ on compact subsets of the plane follows from the uniform bound on the integral of $G^2$. Therefore,

$$u(P) = \sum_{k=1}^{\infty} c_k\psi_k(P)/\lambda_k\,.$$

For $P$ on $C$,

$$f(t) = \sum_{k=1}^{\infty} c_k\psi_k(t)/\lambda_k\,,$$

whereupon

$$c_k/\lambda_k = \int_0^{2\pi} f(t)\, \psi_k(t)\, dt.$$

Since $u_n{}^*$ converges uniformly to $u$, the derivatives of $u_n{}^*$ converge uniformly to the derivatives of $u$, $P$ off $C$, as may be deduced from Eq. (19) replaced by corresponding derivatives.

THEOREM 6. *Under the hypotheses of Theorem* 2, *if* $\{\psi_k\}_1^\infty$ *is a complete orthonormal system of eigenfunctions in* $\mathscr{L}^2$, *then Eq.* (17) *holds for P off C. The series is uniformly convergent in any compact region of the plane not intersecting C. The series may be repeatedly differentiated termwise in such regions.*

*Proof.* If $f \in \mathscr{L}^2$, then

$$f(\tau) = \underset{n\to\infty}{\text{l.i.m.}}\, f_n(\tau),$$

where

$$f_n(\tau) = \sum_1^n a_k \psi_k(\tau),$$

with $a_k$ given by (18). If $\rho$ is the solution in (7) and $h_n$ is the solution of $f_n = -\mathbf{G} h_n$, then $\rho - h_n = \mathbf{G}^{-1}(f_n - f)$ so that $h_n \to \rho$ in $M^*$. Also,

$$h_n(\tau) = \sum_1^n \lambda_k a_k \psi_k(\tau).$$

With

$$u_n^*(P) = -\mathbf{G}(P)\, h_n = \sum_1^n a_k \psi_k(P)$$

and $h_n(\tau)$ in $\mathscr{L}^2$ identified with $\rho_n$ in $M^*$,

$$\begin{aligned} | u_n^*(P) - u(P) | &\leqslant | \rho(-\log | P - Q |) - \rho_n(-\log | P - Q |) | \\ &\leqslant \| \rho - \rho_n \|_{M^*} \| \log | P - Q | \|_W. \end{aligned}$$

if $P$ is off $C$. This last $W$-norm is uniformly bounded for $P$ on compact sets not intersecting $C$, whereupon the uniform convergence of $u_n^*$ to $u$ on such sets follows.

Moreover, $\partial u_n^*/\partial x$ converges uniformly to $-\rho(\partial \log | P - Q |/\partial x)$ on such sets and a standard argument establishes this last expression as $\partial u/\partial x$. The argument may be repeated, establishing the validity of term-by-term differentiation.

**6.** Wolfe [1] proved theorems analogous to Theorems 1 and 2 for the reduced wave equation $\Delta u + k^2 u = 0$ in the exterior of a slit. Using for $G(P, Q)$ the Hankel function $(-\pi i/2)\, H_0^{(1)}(k | P - Q |) = G^{(1)}(P, Q)$, we can prove these theorems for the closed curve $C$. $G^{(1)}(P, Q)$ is of the form

$$G^{(1)}(P, Q) = \log | P - Q | + a(k) | P - Q |^2 \log | P - Q | + J(P, Q),$$

where the derivatives of order two or less of $J$ are continuous. Thus,

$$G^{(1)}(P(t), Q(\tau)) = K(t, \tau) + R^{(1)}(P(t), Q(\tau)),$$

where $K$ is the kernel in (5) and $R^{(1)}$ is continuously differentiable. The corresponding $\mathbf{R}^{(1)}$ is thus a compact operator from $\mathscr{L}^2$ into $W$, and a compact operator from $M^*$ into $\mathscr{L}^2$. Thus Theorems 1 and 2 hold for $\mathbf{G}^{(1)}$ replacing $\mathbf{G}$ and the reduced wave equation replacing the Laplace equation. We will remark on the eigenfunction expansions later.

We shall now suppose that we are dealing with a linear elliptic homogeneous differential equation $Lu = 0$ which has coefficients whose second derivatives are uniformly Holder continuous and has a global fundamental solution of the form

$$F(P, Q) = A(P, Q) \log | P - Q | + B(P, Q) \tag{20}$$

where $A$, $B$ are defined for all points $P$ and $Q$; and $A$ and $B$ are twice continuously differentiable in the coordinates of $P$ and $Q$, with $A(P, P) \neq 0$. We try to represent the solution of the Dirichlet problem for $Lu = 0$ as

$$u(P) = \mathbf{F}(P)\rho, \qquad \rho \in M^*.$$

Now

$$\begin{aligned} &F(P, Q) \\ &\quad = A(P, P) \log | P - Q | + (A(P, Q) - A(P, P)) \log | P - Q | + B(P, Q) \\ &\quad = A(P, P) \log | P - Q | + R^{(2)}(P, Q), \\ &F(P(t), Q(\tau)) \\ &\quad = A(P(t), P(t))\, K(t, \tau) + R^{(3)}(t, \tau). \end{aligned} \tag{21}$$

In order to show the validity of Theorems 1 and 2, it is necessary and sufficient to show that $\mathbf{R}^{(3)}$ is a compact operator from $\mathscr{L}^2$ to $W$ and also from $M^*$ to $\mathscr{L}^2$. However, $R^{(3)}$ is not once continuously differentiable and so some modification of the proofs of Lemmas 3 and 6 is required. It is sufficient to show that if

$$S(P, Q) = [A(P, Q) - A(P, P)] \log | P - Q |, \tag{22}$$

then $S(P(t), Q(\tau))$ is a compact operator.

If $\Omega$ is a bounded set in $\mathscr{L}^2$ and $g \in \Omega$, then $w = \mathbf{S}g \in W$. If in (22), $\log | P - Q |$ is replaced by $\log(| P - Q | + h)$, $h > 0$, then $\mathbf{S}$ is replaced by $\mathbf{S}_h$ and $w_h = \mathbf{S}_h g$. It is then easy to show using Schwarz inequality that the square of the modulus of continuity of $w_h'$ is bounded basically by the modulus of continuity of

$$\int_0^{2\pi} \log^2(| P(t) - Q(\tau)| + h)\, d\tau \, \| g \|^2.$$

The limit relation as $h \to 0$ establishes that the set $\{w' : w = \mathbf{S}g, g \in \Omega\}$ is a uniformly bounded equicontinuous family. The corresponding family of $S\Omega$ is, therefore, relatively sequentially compact in $W$.

If $\Omega$ is a bounded set in $M^*$ and $\sigma \in \Omega$, we consider $b_h = \mathbf{S}_h \sigma$. Now $b_h(t) = \sigma[S_h(P(t), Q(\cdot)]$, and so

$$| b_h(t_1) - b_h(t_2)| \leqslant \| S_h(P(t_1), Q(\cdot)) - S_h(P(t_2), Q(\cdot))\|_W \, \| \sigma \|_{M^*} .$$

The result holds for the limit as $h \to 0$ and so the set $\{b: b = \mathbf{S}\sigma, \sigma \in \Omega\}$ is a uniformly bounded equicontinuous family and is, therefore, relatively sequentially compact in $\mathscr{L}^2$.

The consequence are summed up in the following.

THEOREM 7. *Theorems* 1 *and* 2 *are true for* $\mathbf{F}$ *of* (20) *replacing* $\mathbf{G}$ *and* $Lu = 0$ *replacing* $\Delta u = 0$.

The proof of the existence of the "eigenfunction" expansion for solutions of $Lu = 0$ proceeds as before. Here $\{\psi_n\}$ are the eigenfunctions of $\mathbf{F}$ and $\psi_n(P) = \lambda_n \mathbf{F}(P)\,\psi_n$ .

THEOREM 8. *Theorems* 5 *and* 6 *are valid if* $F(P, Q)$ *replaces* $G(P, Q)$ *and* $Lu = 0$ *replaces* $\Delta u = 0$. (*The hypothesis would then include that* $F(P, Q)$ *has a complete orthonormal system in* $\mathscr{L}^2$ *and that* $\psi_k(P) = \lambda_k \mathbf{F}(P)\,\psi_k$ .) *The series*

$$u(P) = \sum_{k=1}^{\infty} a_k \psi_k(P), \qquad a_k = \int_0^{2\pi} f(t)\,\psi_k(t)\,dt$$

*may be differentiated term by term off* $C$ *as many times as the order of the differentiability of* $F(P, Q)$, $P \neq Q$. *The derived series converge uniformly on compact sets not intersecting* $C$.

**7.** The theorems of the preceding section are valid, as far as they go, for the exterior problem. However, the boundary condition at infinity, if one is required, may not be met. For example, the $u$ of (1) will be bounded at infinity if and only if $\int_0^{2\pi} g(t)\,dt = 0$. Since it is a requirement generally that the solution to the Dirichlet problem for Laplace's equation in the exterior be bounded at infinity, the representation (1) generally cannot be valid.

We shall indicate the modification of (1) required in this case (the case of Laplace's equation). If $\mathbf{G}v = 0$ has only the trivial solution, then there is a unique $w$ in $\mathscr{L}^2$ for which $\mathbf{G}w = -1$. If $-\mathbf{G}\rho = f$, $\rho$ in $M^*$, then $-\mathbf{G}(\rho - kw) = f - k$, where $k$ is a constant. With $\rho_0 = \rho - kw, f_0 = f - k$, then $-\mathbf{G}\rho_0 = f_0$ . $k$ is to be determined so that $\rho_0(1) = 0$.

$$\int_0^{2\pi} f_0(t)\,w(t)\,dt = -\int_0^{2\pi} w(t)(\mathbf{G}\rho_0)\,dt = \rho_0(-\mathbf{G}w) = \rho_0(1).$$

Thus $\rho_0(1) = 0$ if and only if $\int_0^{2\pi} (f - k)w\,dt = 0$, i.e., $k = \int_0^{2\pi} fw\,dt / \int_0^{2\pi} w\,dt$. With this choice of $k$, $\mathbf{G}(P)\,\rho_0 \to 0$ as $|P| \to \infty$. Thus $u(P) = -\mathbf{G}(P)\,\rho_0 + k$ is the function harmonic in the exterior with boundary values $f$.

## References

1. P. Wolfe, "Some Distribution Solutions of Integral Equations and Their Application to Partial Differential Equations," Tech. Rep. 69–133, Math. Dept., Univ. of Md., 1969.
2. S. N. Karp and S. E. Shamma, A generalization of separability in boundary value problems, *SIAM J. Appl. Math.* **20** (1971), 536–546.
3. F. John, "Plane Waves and Spherical Means Applied to Partial Differential Equations," Interscience, New York, 1955.
4. G. Bachman and L. Narici, "Functional Analysis," Academic Press, New York, 1966.
5. C. Miranda, "Partial Differential Equations of Elliptic Type" 2nd rev. ed., Springer–Verlag, New York, 1970.
6. M. Jaswon, Integral equation methods in potential theory, I, *Proc. Roy. Soc. London A* **275** (1963), 23–32.